# Uncertainty Quantification in Linear Interpolation for Isosurface Extraction


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# Isosurface Visualization

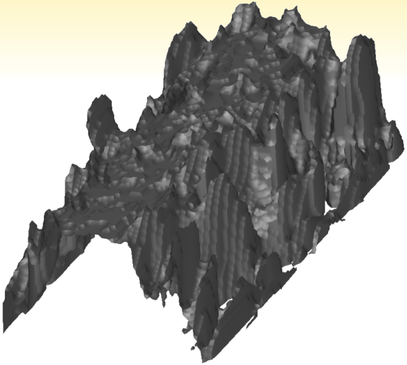
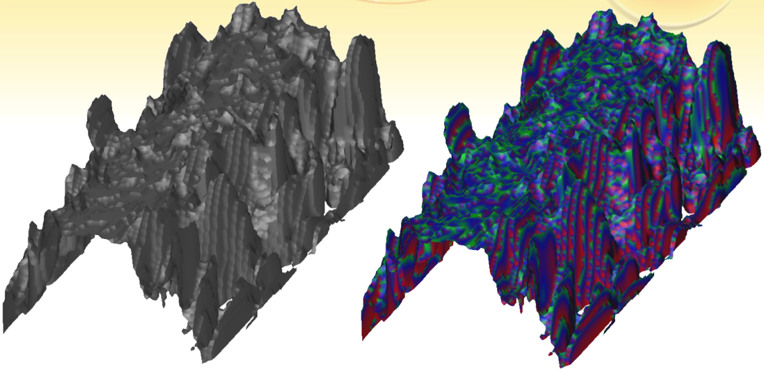


Figure: Isosurface representing the isovalue of $0\,^{\circ}\mathrm{C}$ in a temperature field. Temperature dataset is courtesy of DEMETER project [Palmer et al., 2004]

# Isosurface Extraction from Uncertain Data



(a) Isosurface Visualization (b) Positional Uncertainties

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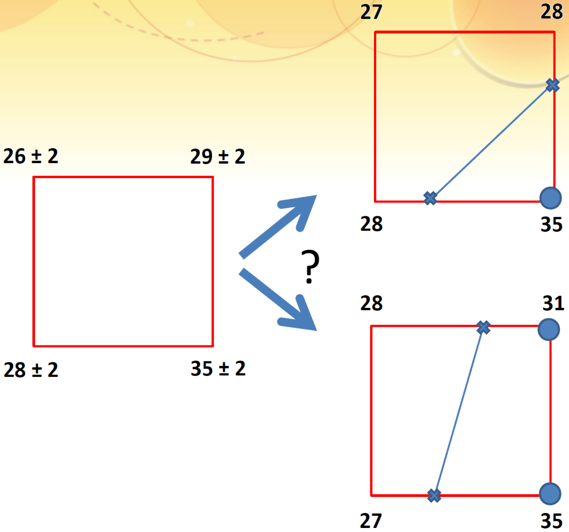
# Uncertainty Visualization

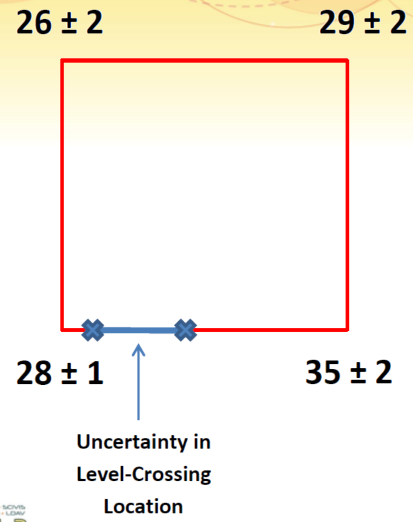
- Uncertainties or errors are introduced in various phases of the visualization pipeline (from data acquisition until the final visualization), e.g., measurement errors.
- Quantification and visualization of the uncertainties - important reasearch direction.
- We study the effect of uncertain data on the marching cubes algorithm (MCA) used for isosurface visualization [Lorensen and Cline, 1987].
    - Cell Configuration Uncertainties.
    - Geometric Uncertainties.

# Cell Configuration Uncertainty




isovalue $= 30$

# Geometric Uncertainty

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# Related Work and Contribution

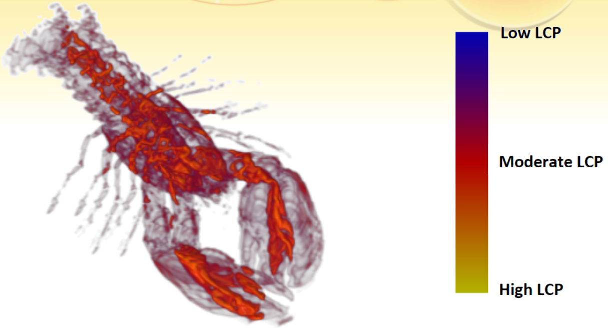
# Uncertainty Visualization Techniques

- Glyphs for flow field uncertainty visualization [Wittenbrink et al., 1996].
- Color and opacity mapping [Rhodes et al., 2003].
- Primitive displacement in the surface normal direction proportional to the uncertainty [Grigoryan and Rheingans, 2004].
- Animation techniques, e.g., animated visual vibrations [Brown, 2004], probabilistic animation [Lundstrom et al., 2007].

# Uncertainty Quantification Techniques

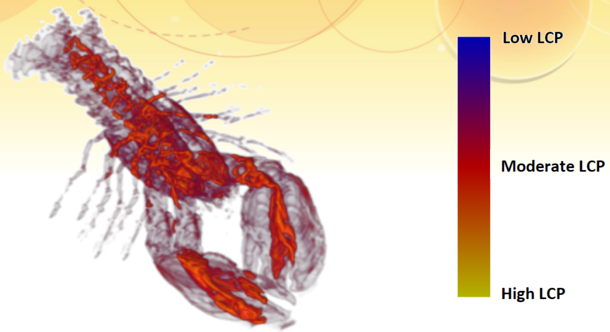
- Isosurface condition analysis to visualize the regions of isosurface, which are sensitive to small data changes [Pöthkow and Hege, 2011].
- Visualization of anisotropic correlation structures to study structural variability in level sets [Pfaffelmoser and Westermann, 2012].
- Choice of Gaussian process regression over trilinear interpolation when data uncertainty is modeled using additive Gaussian noise [Schlegel et al., 2012].

# Probabilistic Marching Cubes [Pöthkow et al., 2011]




Direct volume rendering of the probabilities of the level set crossing the cells, aka, level-crossing probabilities (LCP).

# Probabilistic Marching Cubes [Pöthkow et al., 2011]




- Monte-Carlo sampling from Gaussian, non-parametric distributions [Pöthkow and Hege, 2013].
- Approximate level-crossing probability (LCP) derived using Monte-Carlo approach $= \frac{\#\text{ samples that cross the isosurface}}{\# samples}$.

# Contribution

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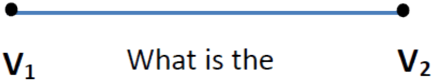


What is the ***probability density function*** of the edge $V_1V_2$ crossed by the isosurface?

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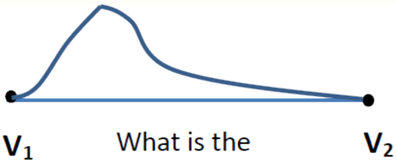


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- Motivated by the work of Pöthkow and Hege, we study the edge-crossing probability density function.
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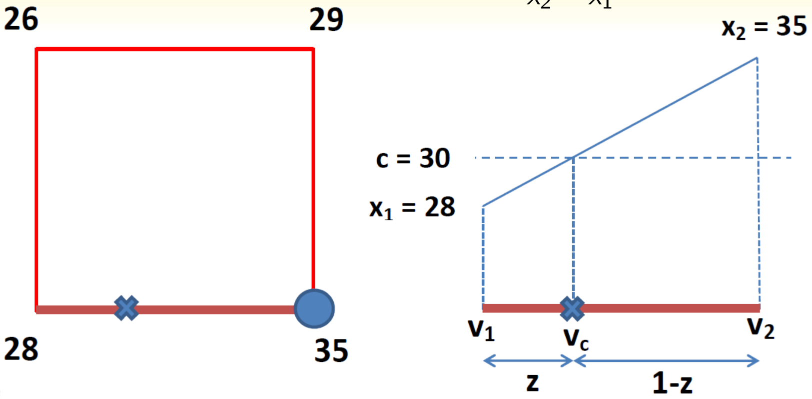
- Motivated by the work of Pöthkow and Hege, we study the edge-crossing probability density function.
- We obtain analytic density function when data uncertainty is modeled using uniform or kernel-based non-parametric distributions.
- Closed-form characterization is efficient.

# Problem Description

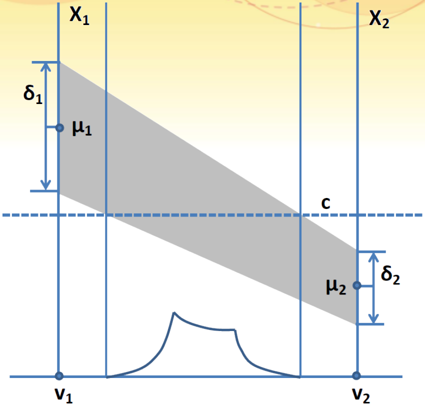
# Inverse Linear Interpolation

The level-crossing location for isovalue c, $\mathbf{v}_c$, is approximated using the *inverse linear interpolation* formula,

$$\mathbf{v}_c = (1-z)\mathbf{v}_1 + z\mathbf{v}_2, \quad \text{where} \quad z = \frac{c - x_1}{x_2 - x_1}.$$



# Uncertainty Quantification in Linear Interpolation




**Aim :** Closed-form characterization of the ratio random variable, $Z = \frac{c - X_1}{X_2 - X_1}$, assuming $X_1$ and $X_2$ have uniform distributions.

$\mu_i$ and $\delta_i$ represent mean and width, respectively, of a random variable $X_i$. c is the isovalue. $\mathrm{v}_1$ and $\mathrm{v}_2$ represent the grid vertices.

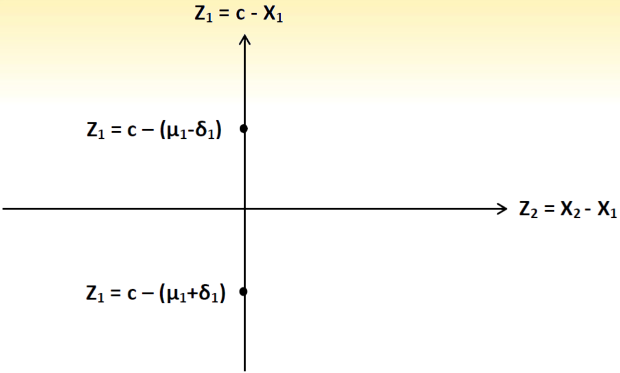
# Approach

# Joint Distribution

Find the joint distribution of the dependent random variables $Z_1 = c - X_1$ and $Z_2 = X_2 - X_1$, where

$$Z = \frac{Z_1}{Z_2} = \frac{c - X_1}{X_2 - X_1}.$$

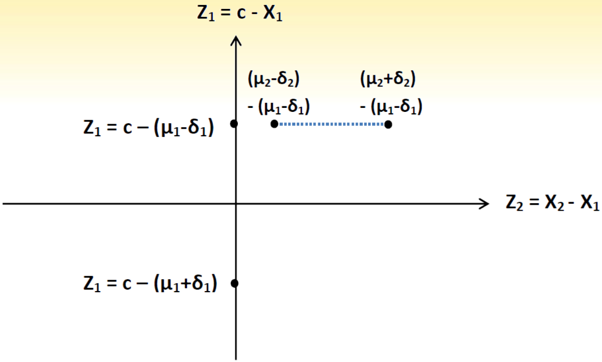
# Joint Distribution




- Determine the range of $c - X_1$.
- $X_1$ assumes values in the range $[\mu_1 - \delta_1, \mu_1 + \delta_1]$.
- Random variables $Z_1$ and $Z_2$ are dependent.

$\mu_i$ and $\delta_i$ represent mean and width, respectively, of a random variable $X_i$.

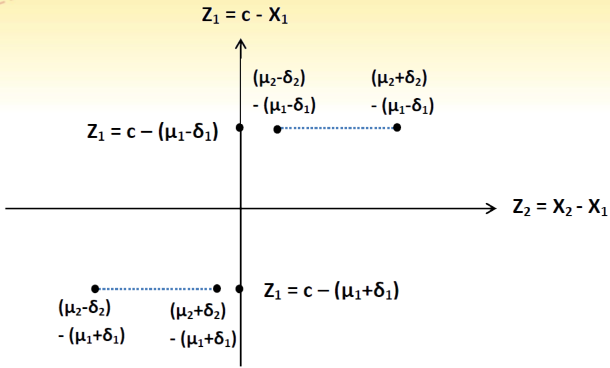
# Joint Distribution




- Determine the range of $X_2 - X_1$.
- $X_2$ assumes values in the range $[\mu_2 - \delta_2, \mu_2 + \delta_2]$.
- Random variables $Z_1$ and $Z_2$ are dependent.

$\mu_i$ and $\delta_i$ represent mean and width, respectively, of a random variable $X_i$.

# Joint Distribution

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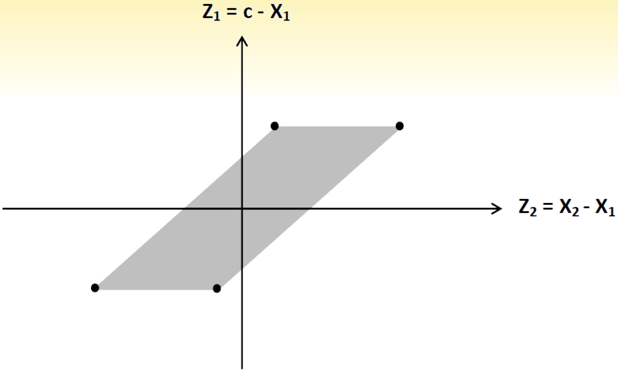
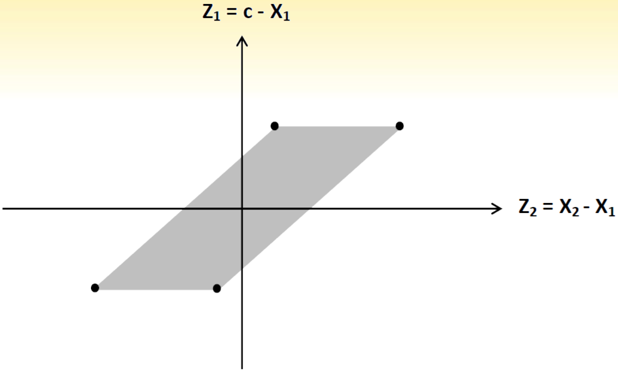



- Parallelogram represents the joint distribution of the dependent random variables $Z_1 = c - X_1$ and $Z_2 = X_2 - X_1$.

# Joint Distribution

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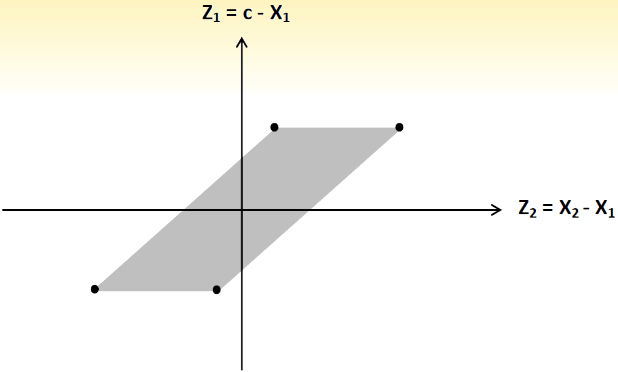
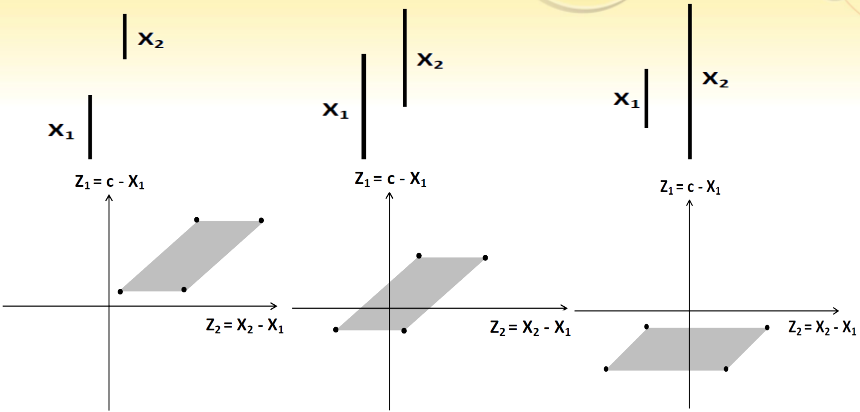
# Joint Distribution




- Parallelogram represents the joint distribution of the dependent random variables $Z_1 = c - X_1$ and $Z_2 = X_2 - X_1$.
- Uniform kernel: parallelogram with constant height.
- Parzen window, triangular kernels: parallelogram with height described by a polynomial function.

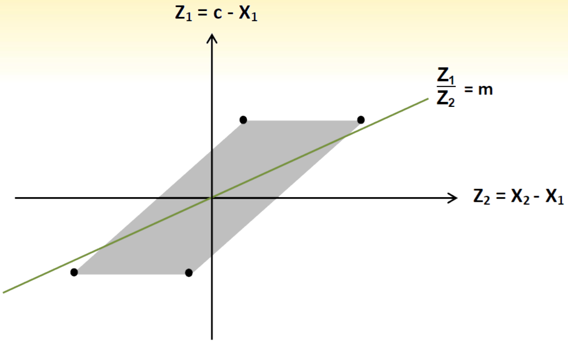
# Joint Distribution

Shape and position of the joint distribution is impacted by the relative configurations for $X_1$ and $X_2$ and the isovalue c.



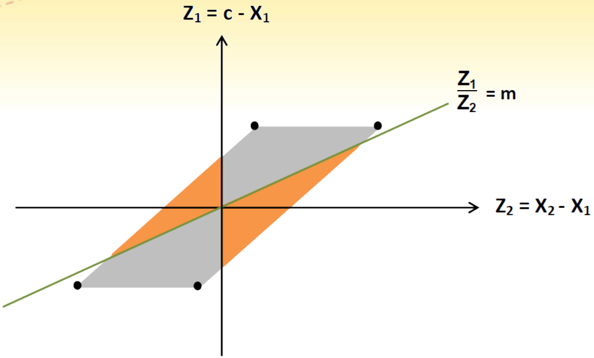

(a) Non-overlapping (b) Overlapping (c) Contained

# Probability Density Function




What is $\Pr(\frac{Z_1}{Z_2} \leq m)$?

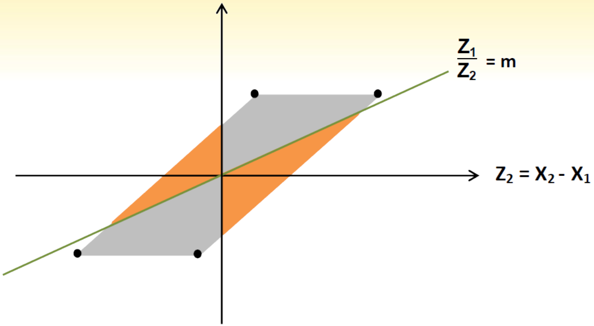
# Probability Density Function




- What is $\Pr(\frac{Z_1}{Z_2} \leq m)$?
- $\mathrm{cdf_Z(m)} = \Pr(-\infty \leq \frac{Z_1}{Z_2} \leq m)$ (orange region).

$\mathrm{cdf_Z(m)}$ represents cumulative density function of a random variable $Z$.

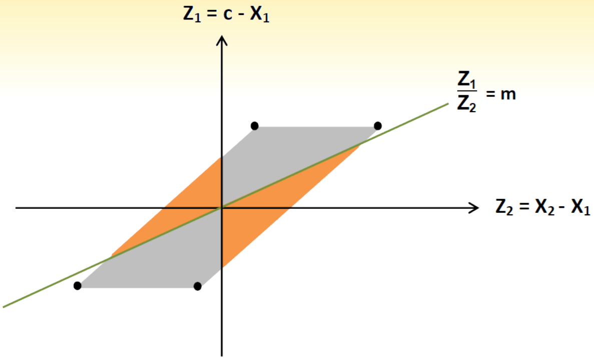
# Probability Density Function




- What is $\Pr(\frac{Z_1}{Z_2} \leq m)$?
- $\mathrm{cdf_Z(m)} = \Pr(-\infty \leq \frac{Z_1}{Z_2} \leq m)$ (orange region).
- Obtain $\mathrm{pdf_Z(m)}$ by differentiating $\mathrm{cdf_Z(m)}$ with respect to m.

$\mathrm{pdf_Z(m)}$ represents probability density function of a random variable $Z$.

# Probability Density Function

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- A piecewise function.

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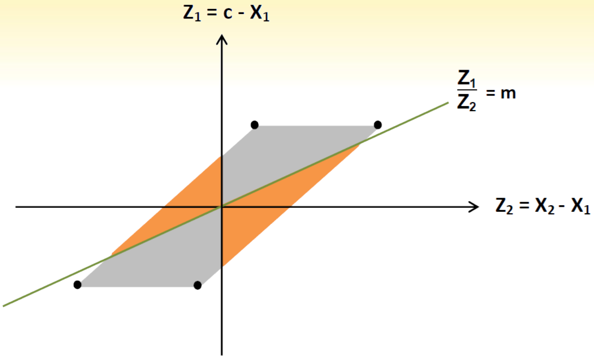


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- A piecewise function.
- Each piece is an inverse polynomial.

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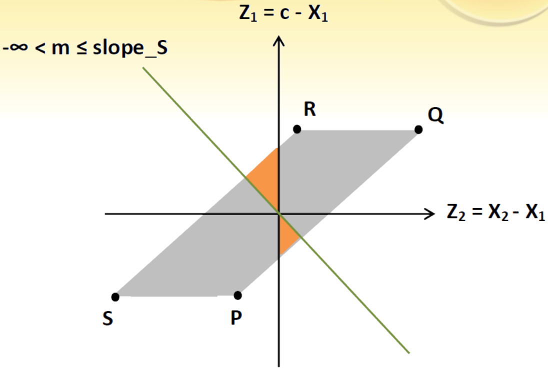
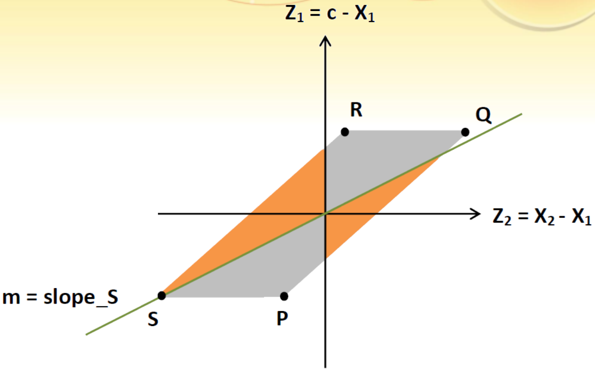



$$pdf_Z(m) = \frac{(c-\mu_2)^2+\delta_2^2}{4\delta_1\delta_2(1-m)^2}$$

# Probability Density Function

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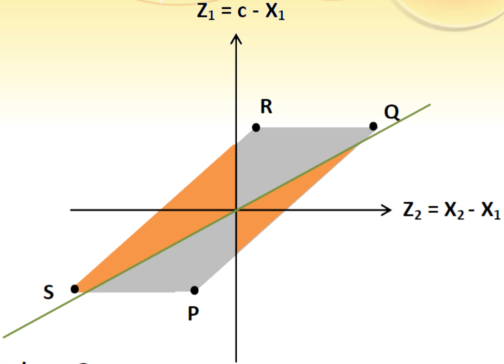


slope_S < m ≤ slope_Q

$$pdf_Z(m) = \frac{(\mu_2+\delta_2-c)^2 m^2+(\mu_1+\delta_1-c)^2(1-m)^2}{8\delta_1\delta_2 m^2(1-m)^2}$$

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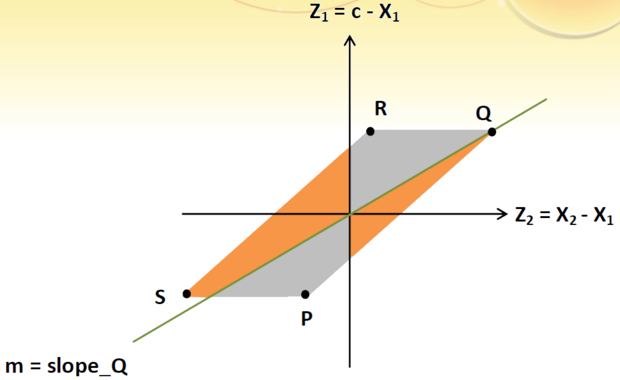


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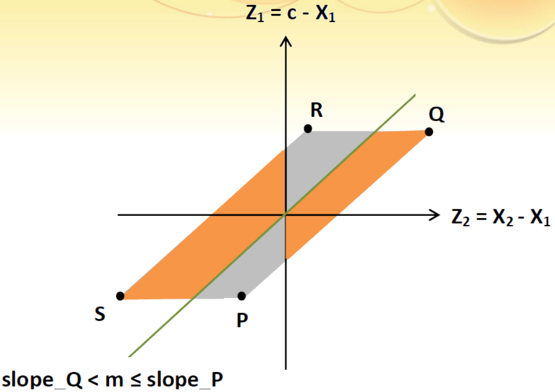


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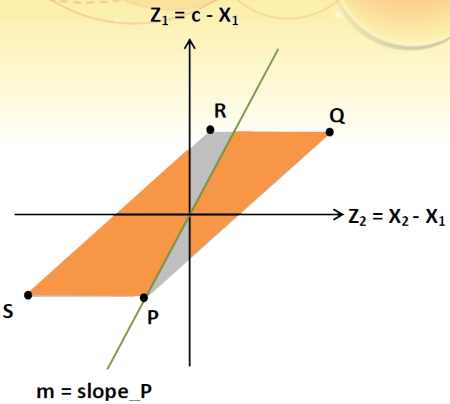


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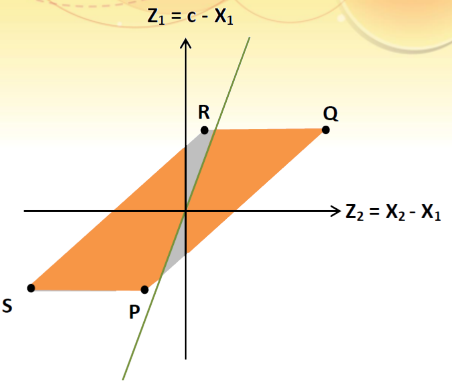


slope_P < m ≤ slope_R

$$pdf_Z(m) = \frac{(\mu_2+\delta_2-c)^2 m^2 + (\mu_1-\delta_1-c)^2 (1-m)^2}{8\delta_1\delta_2 m^2 (1-m)^2}$$

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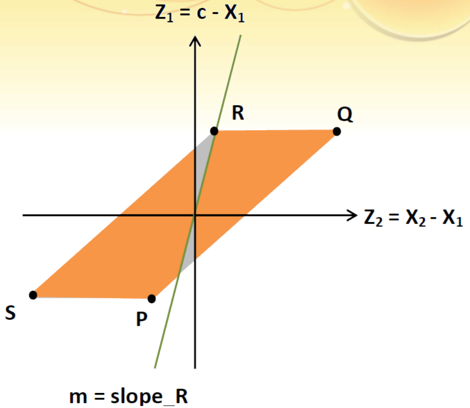


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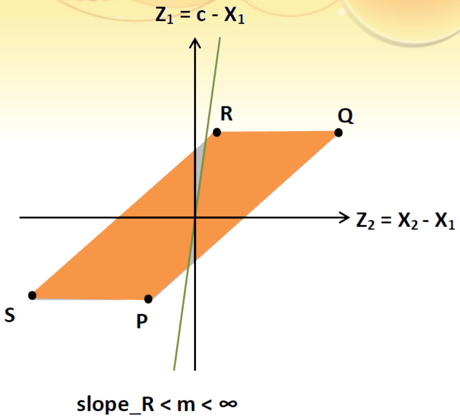



$$pdf_Z(m) = \frac{(c-\mu_2)^2+\delta_2^2}{4\delta_1\delta_2(1-m)^2}$$

# Probability Density Function

We get a piecewise density function as follows, where each piece is an inverse polynomial:

$$pdf_Z(m) = \begin{cases} \frac{(c-\mu_2)^2+\delta_2^2}{4\delta_1\delta_2(1-m)^2}, & -\infty < \mathrm{m} \le slope\_S. \\ \frac{(\mu_2+\delta_2-c)^2m^2+(\mu_1+\delta_1-c)^2(1-m)^2}{8\delta_1\delta_2 m^2(1-m)^2}, & slope\_S < \mathrm{m} \le slope\_Q. \\ \frac{(c-\mu_1)^2+\delta_1^2}{4\delta_1\delta_2 m^2}, & slope\_Q < \mathrm{m} \le slope\_P. \\ \frac{(\mu_2+\delta_2-c)^2m^2+(\mu_1-\delta_1-c)^2(1-m)^2}{8\delta_1\delta_2 m^2(1-m)^2}, & slope\_P < \mathrm{m} \le slope\_R. \\ \frac{(c-\mu_2)^2+\delta_2^2}{4\delta_1\delta_2(1-m)^2}, & slope\_R < \mathrm{m} < \infty. \end{cases}$$

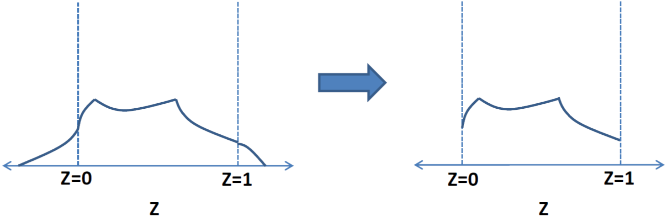
# Application to Marching Cubes Algorithm

# Marching Uncertain Cubes

- Determine cell edges that are getting crossed by the isosurface using MCA [Lorensen and Cline, 1987].

# Marching Uncertain Cubes

- Determine cell edges that are getting crossed by the isosurface using MCA [Lorensen and Cline, 1987].
- Find the edge-crossing density function for the edges that are crossed by the isosurface (conditional probability).



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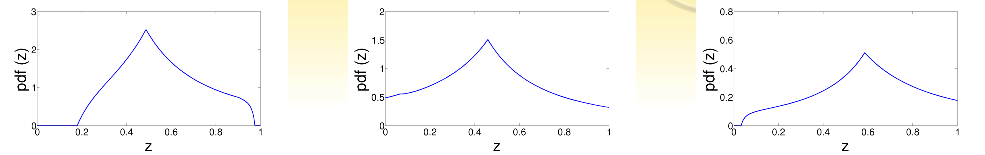


(d) Non-overlapping (e) Overlapping (f) Contained

Figure: Exemplar probability distribution functions, limited to the domain [0,1], for various interval cases. From left to right: non-overlapping ($\mu_1 = 3$, $\delta_1 = 3$, $\mu_2 = 12$, $\delta_2 = 4$, and $c = 7.8$), overlapping ($\mu_1 = 5$, $\delta_1 = 4$, $\mu_2 = 12$, $\delta_2 = 6$, and $c = 8.8$) and contained intervals ($\mu_1 = 7$, $\delta_1 = 2$, $\mu_2 = 8$, $\delta_2 = 6$, and $c = 4.9$).

# Marching Uncertain Cubes

- Determine cell edges that are getting crossed by the isosurface using MCA [Lorensen and Cline, 1987].
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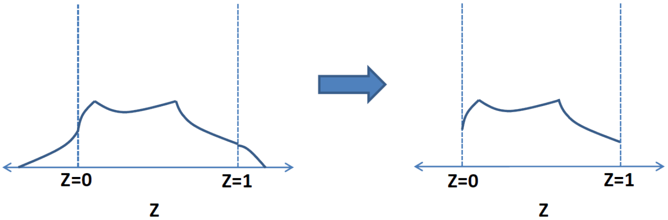


- Find **expected** crossing location and quantify its uncertainty using the **variance**.

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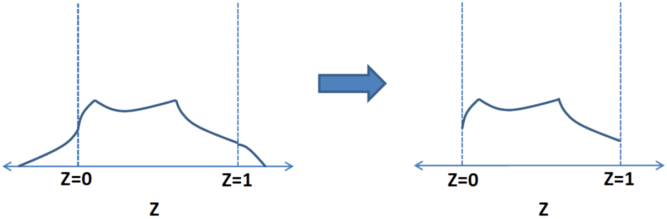
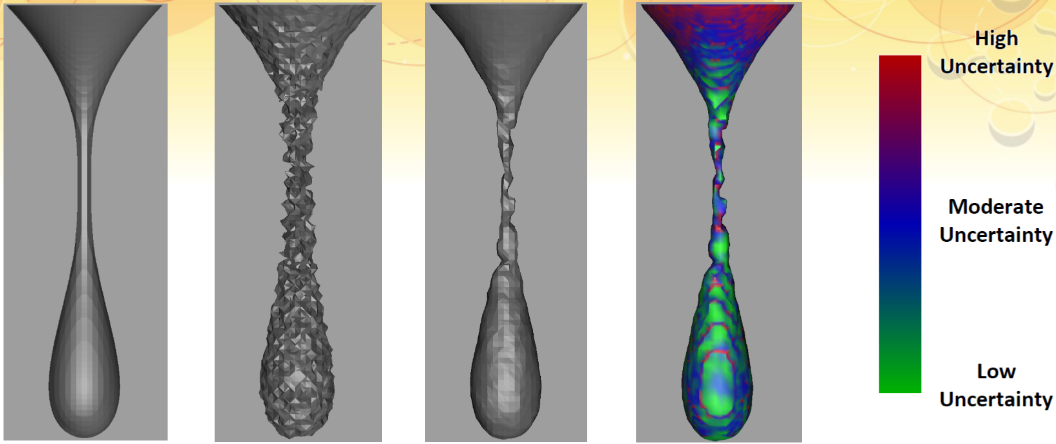



- Find **expected** crossing location and quantify its uncertainty using **variance**.
- Analytic edge-crossing density function allows closed-form computations of the expected value and variance.

# Results

# Expected Crossing and Variance




(a) Ground Truth

(b) Linear Interpolation Isosurface

(c) Expected Isosurface

(d) Positional Uncertainties

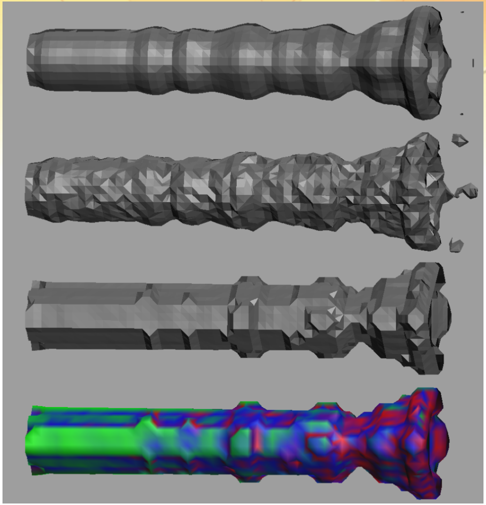
# Expected Crossing and Variance

Ground Truth

Linear Interpolation Isosurface

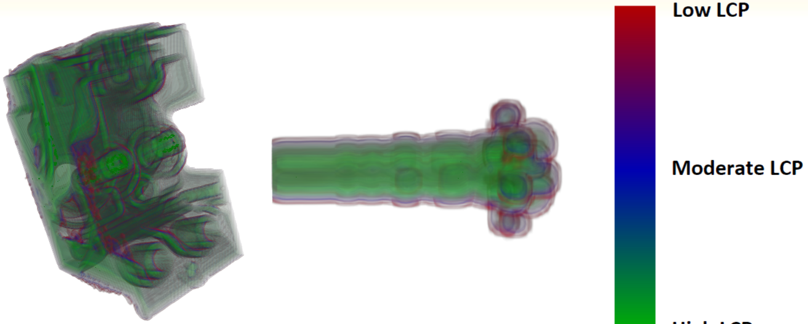
Expected Isosurface

Positional Uncertainties



# Cell Crossing Probabilities

- Level-crossing probabilities (LCP) for each cell can be computed in closed form.
- Volume rendering of the LCP.



# Summary

- Analytic formulation of the edge-crossing probability density function for the kernels with compact support.
- Efficient stable isosurface reconstruction (with closed-form expected value computation) from uncertain data.
- Visualization of the positional uncertainties in the expected isosurface (with closed-form variance computation).

# Thank You for Your Attention!

This research is supported by the AFOSR grant FA9550-12-1-0304, NSF grant CCF-1018149, and ONR grant N000141210862.